ON THE INVARIANCE OF AN OPTIMAL PROCESS WITH DISTRIBUTED PARAMETERS

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We consider the problem of finding a group of transformations G under which an optimal process is invariant. In other words, we find a group G such that the manifold defined by the differential equations describing the controlled process and functional is invariant.

1. Statement of the problem. Let a controlled process be described by the following nonlinear equation in dimensionless form:

$$\frac{\partial\varphi}{\partial t}-\frac{\partial}{\partial x}\left(f(\varphi)\frac{\partial\varphi}{\partial x}\right)+F(u,\varphi)=0\qquad t\in[0,T],\qquad x\in[0,l]\tag{1.1}$$

where $\varphi(t, x)$ is the required distribution, $u(t, x)$ is the disturbed control, and $f(\varphi)$, $F(u, \varphi)$ are continuous functions differentiable sufficiently often with respect to their arguments.

We assume the existence of a distributed control $u(t, x)$ which minimizes the functional

$$J_u=\int_0^T\int_0^l Q(u,\varphi)\,dx\,dt\tag{1.2}$$

Here T, l are known positive constants and $Q(u, \varphi)$ is a continuous function differentiable sufficiently often.

Let us introduce the notion of an invariant optimal process. By this we mean an optimal process which (provided it exists) remains invariant under some group of transformations G. The existence of such an optimal process can therefore be established by constructing such a group of transformations G and finding the relationship among the functions $f(\varphi)$, $F(u, \varphi)$, $Q(u, \varphi)$ such that the manifold Ω defined by the equations

$$\frac{\partial\varphi}{\partial t}-\frac{\partial w}{\partial x}+F(u,\varphi)=0,\qquad w-f(\varphi)\frac{\partial\varphi}{\partial x}=0\tag{1.3}$$

and functional (1.2) remain invariant for any transformation $T_a\in G$. In other words, we are to find transformations of the variables t, x, φ, w, u and of the derivatives of φ, u, w with respect to t and x such that the image points with the coordinates t^*, x^*, φ^*, w^*, u^*, $\varphi^*_{t^*}$, $\varphi^*_{x^*}$, ..., $w^*_{x^*}$ belong to Ω while the value of functional J_u remains unchanged.

2. Construction of the fundamental group G. The group of transformations G is defined by the Lie algebra of the infinitesimal operators

$$Y=\xi_t\frac{\partial}{\partial t}+\xi_x\frac{\partial}{\partial x}+\xi_\varphi\frac{\partial}{\partial\varphi}+\xi_u\frac{\partial}{\partial u}+\xi_w\frac{\partial}{\partial w}\tag{2.1}$$

We introduce the group G^* (the first extension of the G-transformations) defined by the expression

$$Y^*=Y+\xi_{\varphi_t}\frac{\partial}{\partial\varphi_t}+\xi_{\varphi_x}\frac{\partial}{\partial\varphi_x}+\xi_{u_t}\frac{\partial}{\partial u_t}+\xi_{u_x}\frac{\partial}{\partial u_x}+\xi_{w_t}\frac{\partial}{\partial w_t}+\xi_{w_x}\frac{\partial}{\partial w_x}\tag{2.2}$$

where G^* is isomorphic to G.

The condition of invariance of functional (1.2) can be written as

$$Y[J_u]=0\tag{2.3}$$

i.e. the values of the functional remain unchanged if the variables t, x, φ, u, w are subjected to the transformation

$$t^* = t + \varepsilon\xi_t, \quad x^* = x + \varepsilon\xi_x, \quad u^* = u + \varepsilon\xi_u, \quad \varphi^* = \varphi + \varepsilon\xi_\varphi, \quad w^* = w + \varepsilon\xi_w$$

Here $\xi_t, \xi_x, \xi_\varphi, \xi_u, \xi_w$ are the coordinates of the operator Y which are functions of the coordinates of the space E_5; ε is a small parameter.

As in [1], we have the following relation for $T_a \in G$:

$$T_a J_u = \int_0^T \int_0^l Q(u + \varepsilon\xi_u, \varphi + \varepsilon\xi_\varphi)\, d(x + \varepsilon\xi_x)\, d(t + \varepsilon\xi_t) = \int_0^T \int_0^l [Q(u, \varphi) + \varepsilon Y(Q)] \times$$

$$\times \left[1 + \varepsilon\left(\frac{\partial\xi_x}{\partial x} + \frac{\partial\xi_t}{\partial t}\right)\right] dx\, dt = \int_0^T \int_0^l \left[Q(u, \varphi) + \varepsilon\left(Y(Q) + Q(u, \varphi)\left(\frac{\partial\xi_x}{\partial x} + \frac{\partial\xi_t}{\partial t}\right)\right)\right] dx\, dt$$

Functional (1.2) is invariant if and only if

$$Y[Q(u, \varphi)] + Q(u, \varphi)\left(\frac{\partial\xi_x}{\partial x} + \frac{\partial\xi_t}{\partial t}\right) = 0 \tag{2.4}$$

The conditions of invariance of the manifold Ω defined by system (1.3) are of the form [2]

$$Y^*[\Omega] = 0 \tag{2.5}$$

The conditions of invariance of the functional J_u and of the manifold Ω, give us the system of defining equations of the Lie algebra. This system enables us to determine the coordinates of the infinitesimal operator Y and to establish relationship among the functions $f(\varphi)$, $F(u, \varphi)$, $Q(u, \varphi)$. These conditions are

$$\begin{gathered}
\frac{\partial Q}{\partial u}\xi_u + \frac{\partial Q}{\partial \varphi}\xi_\varphi + Q(u, \varphi)\left(\frac{\partial\xi_x}{\partial x} + \frac{\partial\xi_t}{\partial t}\right) = 0 \\
\xi_{\varphi_t} - \xi_{w_x} + \frac{\partial F}{\partial u}\xi_u + \frac{\partial F}{\partial \varphi}\xi_\varphi = 0 \\
f(\varphi)\xi_{\varphi_x} + \frac{df}{\partial\varphi}\frac{\partial\varphi}{\partial x}\xi_\varphi - \xi_w = 0
\end{gathered} \tag{2.6}$$

Investigation of the defining equations for the coordinates of the infinitesimal operator Y yields the following relations:

$$\xi_\varphi = \left(2\frac{\partial\xi_x}{\partial x} - \frac{d\xi_t}{dt}\right)\frac{f}{f'} \tag{2.7}$$

$$\xi_w = \left\{\left[1 + 2\left(\frac{f}{f'}\right)'\right]\frac{\partial\xi_x}{\partial x} - \left[1 + \left(\frac{f}{f'}\right)'\right]\frac{d\xi_t}{dt}\right\}w + 2\frac{f^2}{f'}\frac{\partial^2\xi_x}{\partial x^2} \tag{2.8}$$

$$\begin{aligned}
\xi_u = \frac{1}{\partial F/\partial u}\Bigg\{&\left[F\left(\frac{f}{f'}\right)' - \frac{\partial F}{\partial\varphi}\frac{f}{f'}\right]\left(2\frac{\partial\xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) - \\
&- \frac{f}{f'}\left(2\frac{\partial^2\xi_x}{\partial x\,\partial t} - \frac{d^2\xi_t}{dt^2}\right) + 2\frac{f^2}{f'}\frac{\partial^3\xi_x}{\partial x^3} - F\frac{d\xi_t}{dt}\Bigg\}
\end{aligned} \tag{2.9}$$

$$\left(\frac{f}{f'}\right)''\left(2\frac{\partial\xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) = 0, \quad \left[2\frac{1}{f}\left(\frac{f^2}{f'}\right)' + 1 + 2\left(\frac{f}{f'}\right)'\right]\frac{\partial^3\xi_x}{\partial x^3} = 0 \tag{2.10}$$

$$\begin{aligned}
&\left\{\frac{\partial Q}{\partial\varphi}\frac{f}{f'} + \frac{\partial Q/\partial u}{\partial F/\partial u}\left[F\left(\frac{f}{f'}\right)' - \frac{f}{f'}\frac{\partial F}{\partial\varphi}\right]\right\}\left(2\frac{\partial\xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) - \\
&- \frac{\partial Q/\partial u}{\partial F/\partial u}\left[\frac{f}{f'}\left(2\frac{\partial^2\xi_x}{\partial x\,\partial t} - \frac{d^2\xi_t}{dt^2}\right) - 2\frac{f^2}{f'}\frac{\partial^3\xi_x}{\partial x^3} + F\frac{d\xi_t}{dt}\right] + Q(u, \varphi)\left(\frac{\partial\xi_x}{\partial x} + \frac{d\xi_t}{dt}\right) = 0
\end{aligned} \tag{2.11}$$

A. Let us determine the group of transformations G for an arbitrary relationship among $f(\varphi)$, $F(u, \varphi)$, $Q(u, \varphi)$.

From (2.10) and (2.11) we find that

$$\frac{\partial \xi_t}{dt} = \frac{\partial \xi_x}{\partial x} = \frac{\partial^2 \xi_x}{\partial t\, \partial x} = \frac{\partial^2 \xi_x}{\partial x^2} = \frac{\partial^3 \xi_x}{\partial x^3} = 0$$

Then

$$\xi_\varphi = \xi_u = \xi_w = 0, \qquad \xi_t = a_1, \ \xi_x = a_2 \tag{2.12}$$

Hence, the basis of the Lie algebra of the fundamental group of system (1.3) and functional (1.2) consists of the operators

$$Y_1 = \partial(\cdot)/\partial t, \ \ Y_2 = \partial(\cdot)/\partial x \tag{2.13}$$

B. As noted in [2], the group of transformations G can be extended by way of a special form of the function $f(\varphi)$,

$$f(\varphi) = c_1 \varphi^{2m}, \ f(\varphi) = c_2 e^{n\varphi} \ (c_1, c_2, \ m, \ n = \mathrm{const})$$

Let us consider the problem of finding the group of transformations G under which the optimal process is invariant for $f(\varphi) = c_1 \varphi^{2m}$ and $m \neq -{}^2/_3$.

Relations (2.7) – (2.11) here become

$$\xi_\varphi = \frac{1}{2m}\left(2\frac{\partial \xi_x}{\partial x} - \frac{d\xi_t}{dt}\right)\varphi, \qquad \xi_w = \frac{1}{m}\left[(m+1)\frac{\partial \xi_x}{\partial x} - (2m+1)\frac{d\xi_t}{dt}\right]w$$

$$\xi_u = \frac{1}{\partial F/\partial u}\left[\frac{1}{2m}\left(F - \varphi\frac{\partial F}{\partial \varphi}\right)\left(2\frac{\partial \xi_x}{\partial x} - \frac{d\xi_t}{dt}\right) - \frac{1}{2m}\left(2\frac{\partial^2 \xi_x}{\partial t\,\partial x} - \frac{d^2\xi_t}{dt^2}\right)\varphi - F\frac{d\xi_t}{dt}\right] \tag{2.14}$$

$$\partial^2 \xi_x / \partial x^2 = 0 \tag{2.15}$$

$$\left[\varphi\frac{\partial Q}{\partial \varphi} + \frac{\partial Q/\partial u}{\partial F/\partial u}\left(F - \varphi\frac{\partial F}{\partial \varphi}\right) + mQ\right]\frac{\partial \xi_x}{\partial x} - \left\{\frac{\varphi}{2}\frac{\partial Q}{\partial \varphi} + \right.$$

$$\left. + \frac{1}{2}\frac{\partial Q/\partial u}{\partial F/\partial u}\left[(2m+1)F - \varphi\frac{\partial F}{\partial \varphi}\right] - mQ\right\}\frac{d\xi_t}{dt} -$$

$$- \frac{1}{2}\frac{\partial Q/\partial u}{\partial F/\partial u}\left(2\frac{\partial^2 \xi_x}{\partial t\,\partial x} - \frac{d^2 \xi_t}{dt^2}\right)\varphi = 0 \tag{2.16}$$

Since $F = F(u, \varphi)$, $Q = Q(u, \varphi)$, it follows from (2.16) with allowance for (2.15) that the following relations are possible among the coordinates of the infinitesimal operators defining the group of transformations G under which the optimal process is invariant for $f = c_1 \varphi^{2m}$:

1°.

$$\begin{gathered}\xi_x = \alpha_1 x + \alpha_0, \quad \xi_t = \alpha_1 t + \beta_0, \quad \xi_\varphi = \frac{1}{2m}\alpha_1 \varphi \\ \xi_w' = -\alpha_1 w, \quad \xi_u = \frac{1}{2m}\left[(1-2m)F - \varphi\frac{\partial F}{\partial \varphi}\right]\frac{\alpha_1}{\partial F/\partial u}\end{gathered} \tag{2.17}$$

Here α_1, α_0, β_0 are the defining constants. Hence, the basis of the Lie algebra of the fundamental group consists of the operators

$$Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial x}, \qquad Y_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} +$$

$$+ \frac{1}{2m}\varphi\frac{\partial}{\partial \varphi} - w\frac{\partial}{\partial w} + \frac{1}{2m}\left[(1-2m)F - \varphi\frac{\partial F}{\partial \varphi}\right]\frac{1}{\partial F/\partial u}\frac{\partial}{\partial u} \tag{2.18}$$

The functions $Q(u, \varphi)$, $F(u, \varphi)$ are related by the expression

$$\varphi \frac{\partial Q}{\partial \varphi} + \frac{1}{\partial F / \partial u}\left[(1-2m) F - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{\partial Q}{\partial u} + 4mQ = 0$$

2°
$$\xi_x = \alpha_1 e^t x + \alpha_0, \quad \xi_t = \alpha_1 e^t + \beta_0, \quad \xi_\varphi = \frac{\alpha_1}{2m} e^t \varphi \tag{2.19}$$

$$\xi_w = -\alpha_1 e^t w, \quad \xi_u = \frac{1}{2m} e^t \left[(1-2m) F - \varphi - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{\alpha_1}{\partial F / \partial u}$$

The basis of the Lie algebra of the fundamental group in this case is

$$Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial x}$$

$$Y_3 = e^t \left\{\frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{1}{2m} \varphi \frac{\partial}{\partial \varphi} - w \frac{\partial}{\partial w} + \frac{1}{2m}\left[(1-2m) F - \varphi - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{1}{\partial F / \partial u} \frac{\partial}{\partial u}\right\}$$

and $Q(u, \varphi)$, $F(u, \varphi)$ are related by the expression

$$\varphi \frac{\partial Q}{\partial \varphi} + \frac{1}{\partial F / \partial u}\left[(1-2m) F - \varphi - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{\partial Q}{\partial u} + 4mQ = 0$$

3°
$$\xi_x = \alpha_0, \quad \xi_t = \beta_1 t + \beta_0, \quad \xi_\varphi = -\frac{\beta_1}{2m} \varphi \tag{2.20}$$

$$\xi_w = -\frac{2m+1}{m} \beta_1 w, \quad \xi_u = -\frac{\beta_1}{2m}\left[(1+2m) F - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{1}{\partial F / \partial u}$$

The basis of the Lie algebra of the fundamental group then consists of the operators

$$Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial x}, \qquad Y_3 = t \frac{\partial}{\partial t} - \frac{1}{2m} \varphi \frac{\partial}{\partial \varphi} -$$
$$- \frac{2m+1}{m} w \frac{\partial}{\partial w} - \frac{1}{2m}\left[(1+2m) F - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{1}{\partial F / \partial u} \frac{\partial}{\partial u}$$

and $Q(u, \varphi)$, $F(u, \varphi)$ must satisfy the relation

$$\varphi \frac{\partial Q}{\partial \varphi} + \frac{1}{\partial F / \partial u}\left[(1+2m) F - \varphi \frac{\partial F}{\partial \varphi}\right] \frac{\partial Q}{\partial u} - 2mQ = 0$$

4°
$$\xi_x = \alpha_1 x + a_0, \quad \xi_t = \beta_0, \quad \xi_\varphi = \frac{\alpha_1}{m} \varphi \tag{2.21}$$

$$\xi_w = \frac{m+1}{m} \alpha_1 w, \quad \xi_u = \frac{\alpha_1}{m}\left(F - \varphi \frac{\partial F}{\partial \varphi}\right) \frac{1}{\partial F / \partial u}$$

Then

$$Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial x}, \qquad Y_3 = x \frac{\partial}{\partial x} + \frac{1}{m} \varphi \frac{\partial}{\partial \varphi} +$$
$$+ \frac{m+1}{m} w \frac{\partial}{\partial w} + \frac{1}{m}\left(F - \varphi \frac{\partial F}{\partial \varphi}\right) \frac{1}{\partial F / \partial u} \frac{\partial}{\partial u}$$

The relationship between $Q(u, \varphi)$ and $F(u, \varphi)$ in this case is defined by the expression

$$\varphi \frac{\partial Q}{\partial \varphi} + \frac{1}{\partial F / \partial u}\left(F - \varphi \frac{\partial F}{\partial \varphi}\right) \frac{\partial Q}{\partial u} + mQ = 0$$

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